ISSN: 0067-3064

DECOMPOSITION OF BICYCLIC GRAPHS INTO PATHS AND CYCLES SUBGRAPH FOR EASY OPTIMIZATION OF NETWORK CONFLICTS

Venkat Narayanan G [1*], Saravanan M [2], Sankar C [3]

[1*] Department of Mathematics, St. Joseph's College of Engineering, Chennai, Tamil Nadu, India. email : gvenkatnarayanan@gmail.com.

[2] PG and Department of Mathematics Research, Mannar Thirumalai Naciker College, Madurai, Tamil Nadu, India. email : msaran81@gmail.com.

[3] Department of Mathematics, St. Joseph's College of Engineering, Chennai, Tamil Nadu, India. email : csankar26@gmail.com.

ABSTRACT

Transport systems are crucial for connectivity and economic growth but face challenges like traffic congestion, pollution, high maintenance costs, and disruptions. A 2-simple graphoidal cover helps simplify these systems by dividing them into smaller components for efficient rerouting, though it can be computationally demanding in dense networks. Bicyclic graphs improve resilience by providing redundancy and fault tolerance, ensuring smooth flow during disruptions. When combined, these tools enhance efficiency, scalability, and distributed management, supporting seamless expansion of transportation networks. This paper examines the decomposition of three types of bicyclic graphs using 2-simple graphoidal covers.

Keywords: Graphoidal Cover, 2-Simple Graphoidal Covers, Bicyclic Graphs, Decomposition Of Graphs, Simple Graphoidal Covers.

AMS Subject Classification: 05C70,05C38

1. Introduction

Graph theory is a prominent tool in computer science engineering and its applications, such as social networks, artificial intelligence-based search engines, disease control, logistics, and many other interrelated domains.

Traffic control on urban roads requires optimizing signal times at junctions to maximize traffic flow. Minimal mobile traffic flow is expected to result in optimal traffic flow. A graph theory model can be used to represent signals as vertices and junctions as edges. To optimize continuous traffic flow, routes that always provide a green signal can be found through graph decomposition, reducing travel time, fuel consumption, and transportation costs. Graph decomposition, as one of the most active areas of graph theory, can be utilized in a wide range of domains, including networking, web search, communicable disease transmission, and contact tracing.

The decomposition of a graph H is a collection of its edge-disjoint sub-graphs since every edge of H lies in exactly one member of the collection. Many authors defined different types of decomposition by applying various parameters.

The Panama Canal serves as a perfect real-world example to understand certain concepts in graph theory, particularly the idea of graphoidal covers. Imagine a map where harbors are dots (vertices) and shipping routes are lines (edges) connecting these dots. The Panama Canal is one such critical edge, connecting harbors in the Atlantic Ocean to those in the Pacific Ocean.

In a traditional simple graphoidal cover, each vertex (harbour) can only be an internal point in one path. Considering if a ship travels from the Atlantic to the Pacific via the Panama Canal, it will be unable to return under this rule. Instead, it would need to find another route, possibly around South America. This restriction creates several problems, like increased distance travelled, higher fuel consumption, extended travel time, and greater transportation costs.

This is where the concept of a 2-simple graphoidal cover becomes valuable. It allows each vertex to be internal in up to two paths instead of just one. (i.e., considering a two-way street versus a one-way street. In practical terms, this means ships can use the Panama Canal in both directions, the same harbor can be part of two different shipping routes, and more efficient routes become possible. The 2-simple graphoidal cover essentially turns the Panama Canal into a two-way street, allowing ships to use it multiple times in their journey. This mathematical concept has practical implications. By allowing vertices (harbors) to be internal points in two paths, we have ships that can take shorter, more direct routes; fuel consumption is optimized, travel time is reduced, and transportation costs are minimized.

2. Literature Review

In 1987, Acharya and Sampath Kumar [1, 2, 14] introduced the graphoidal cover as a close variant of another emerging discrete structure, semigraphs [5]. In their first work, they defined label graphoidal graphs and demonstrated that Beineke's forbidden graphs are graphoidal. Arumugam and Pakkiam [3, 4, 16] conducted extensive research on graphoidal covering numbers (η) for various standard graph families, including complete graphs, wheels, bipartite graphs, trees, and unicyclic graphs.

Arumugam and Suseela [18] investigated whether acyclic graphoidal graphs evolve solely by allowing only open paths in the graphoidal cover. They discovered the acyclic graphoidal covering number (η_a) for unicyclic graphs, as well as the relationship between the acyclic graphoidal covering number and the path partition number (π) for any graph $\Delta \leq 3$. Arumugam, Indra Rajasingh, and Roushini Leely Pushpam [15] describe the class of graphs H for which $\eta_a = \Delta - 1$. Later, Indra Rajasingh and Roushini Leely Pushpam [9] discovered the relationship between graphoidal cover and acyclic graphoidal covers.

Arumugam and Shahul Hamid [8, 17] pioneered the theory of simple graphoidal cover (η_s) by restricting any two paths in graphoidal cover have at most one common vertex. They discovered a relationship between the simple graphoidal coverage number, girth (g), and diameter of the graph (d). Nagarajan et al. [10] were the first to introduce and describe the concept of 2-graphoidal path cover. They [11] extended their findings to m-graphoidal path cover.

Motivation of m graphoidal path cover, the concept of 2-graphoidal cover was started and discussed by Das and Ratan Singh [13]. Ratan Singh and Das [12] and Venkat Narayanan et.al [19] found η, η_a, η_2, η_{2a} and η_{2as} on bicyclic graphs. In this paper, the parameter of 2-simple graphoidal cover is initiated and the value of the parameter of families of bicyclic graphs is determined.

3. Basic Terminology and Results

All of the graphs H = (V, E) in this paper are non-trivial, connected, simple and undirected. The order and size of the graph H are denoted by p and q respectively. Harary [7] can be referred for graph terminology. If $P = (x_0, x_1, ..., x_n)$, then the vertices x_0, x_n are called external vertices and x_1, x_2, ..., x_{n-1} are called internal vertices. If two paths P_1 and P_2 of H have no common internal vertex, then they are internally disjoint.

Definition 3.1. [1] A graphoidal cover (gc) of H is a set ψ_a of open (closed) paths in H satisfying the following conditions.

(C1) At least two vertices in every path in ψ_a.

(C2) Every vertex of H is an internal vertex of at most one path in ψ_a.

(C3) Every edge of H is in exactly one path in ψ_a.

The minimum cardinality of a graphoidal cover (gc) of H is called the graphoidal covering number of H and is denoted by $\eta(H)$.

 ISSN: 0067-3064

Definition 3.2. [20] A 2-simple graphoidal cover (2-simple gc) of H is a set ψ_a of paths in H satisfying the following conditions.

(C1) Every vertex of H is an internal vertex of at most two paths in ψ_a.

(C2) Every edge of H is in exactly one path in ψ_a.

(C3) Any two paths in ψ_a have at most one common vertex.

The minimum cardinality of a 2-simple graphoiodal cover of H is called 2-simple graphoidal covering number of H and is denoted by $\eta_{2s}(H)$.

Remark 3.1. In a 2-simple graphoidal cover (2-simple gc) of G, t_1 denotes total number of internal vertices that appear exactly once in paths of ψ_a. Similarly, t_2 denotes the total number of internal vertices that appear exactly twice in paths of ψ_a, whereas t denotes the total number of external vertices in the paths of ψ_a.

Theorem 3.1. [20] For any (p, q) graph, $\eta_{2s}(H) = q - p - t_2 + t$.

Theorem 3.2. [20] Let H be a tree with n pendant vertices, then $\eta_{2s}(H) = n - 1 - m$, where m is the total number of vertices of degree ≥ 4 in H.

Theorem 3.3. [20] Let H be a unicyclic graph with s pendant vertices. Let C_u be the unique cycle on H. Let L be the number of vertices of degree > 2 on C_u. Then

$$\eta_{2s}(H) = \begin{cases} 1 & if\ \ L = 0 \\ s - r & \begin{array}{l} if\,(L = 1\ and\ \deg(u) \geq 6, where\ u\ is\ the\ only \\ vertex\ of\ \deg > 2\ on\ C_u)(Or)(L = 2,\ u\ and\ v \\ are\ the\ only\ vertices\ of\ \deg > 2\, on\ C_u\ and\ either \\ \deg(u) \geq 6)\ (Or)\deg(v) \geq 6)(Or)(L > 2) \end{array} \\ (s+1) - r & otherwise \end{cases}$$

Where r is the total number of vertices of $\deg \geq 4$ on H.

A simple connected graph H is said to be a bicyclic graph, if it contains exactly two cycles with $q = p + 1$. Bicyclic graphs are commonly used in theoretical chemistry to establish the relationships between the structure and the properties of molecules. They have a relationship with the physical, chemical, and thermodynamic properties of chemical compounds. In this paper we discuss the three kinds of bicyclic graphs. Let B be a bicyclic graph with two cycles $C_{m1} = (x_0, x_1, ..., x_{m-1}, x_0)$ and $C_{m2} = (y_0, y_1, ..., y_{n-1}, y_0)$ with common vertices l ($l > 0$). The vertices in cycle C_{m1} are labelled clockwise, while the vertices in cycle C_{m2} are labelled anti-clockwise.

 ISSN: 0067-3064

a. First type of Bicyclic graph $B_1(m_1, m_2)$ is obtained by two cycles C_{m1} and C_{m2}, share exactly one common vertex $x_0(y_0)$ in Figure 1.

b. Second type of Bicyclic graph $B_2(m_1, m_2, r)$ is obtained by connecting two cycles C_{m1} and C_{m2} by a unique path of length r (r > 1) with starting vertex x_1 and terminal vertex y_1 in Figure 2.

c. Third type of Bicyclic graph $B_3(m_1, m_2, t)$ is obtained by three internal disjoint paths $\{P_1, P_2, P_3\}$ from a vertex u to a vertex v. The paths are $P_1 = (u, x_1, .., x_{m-1}, v)$ with length m, $P_2 = (u, y_1, ..., y_{n-1}, v)$ with length n and $P_3 = (u, w_1, ..., w_{t-1}, v)$ with length $t(1 \le t \le \min\{m, n\})$ in Figure 3.

4. Main Results

Theorem 4.1. Let H be a bicyclic graph with s pendant vertices. Also let $B_1(m_1, m_2)$ be the unique bicycle in H and let j be the number of vertices of degree >2 on $B_1(m_1, m_2)$. Then

$$\eta_{2s}(H) = \begin{cases} 2 & H = B_1(m_1, m_2) \\ (s+3)-r & \begin{aligned} &if\ [(j=1\ and\ \deg(y_0)=5)(Or)(j=2, \deg(y_0)\le 5, \\ &\deg(x_i)\le 5)(Or)(j=3, and\ ((\deg(y_0)=4, \deg(x_i)\le 5, \\ &\deg(x_j)\le 5, x_i, x_j \in C_{m1})(Or)(\deg(y_0)\le 5, \deg(x_i)\le 5, \\ &\deg(x_j)\le 5, x_i \in C_{m1}, x_j \in C_{m2})] \end{aligned} \\ (s+2)-r & \begin{aligned} &if\ [(j=1\ and\ \deg(y_0)=6(or)7)(or)(j=2,\ and \\ &(\deg(y_0)=6(or)7, \deg(x_i)\le 5)(or)(\deg(y_0)\le 5, \\ &\deg(x_i)\ge 6))(Or)(j=3, and\ ((\deg(y_0)=5, \deg(x_i)\ge 3 \\ &\deg(x_j)\ge 3)(or)(either\ (\deg(x_i)\ge 6(or)(\deg(x_j)\ge 6, \\ &\deg(y_0)\le 5)\, x_i, x_j \in C_{m1})(or)(\deg(x_i)\le 5, \deg(x_j)\le 5, \\ &\deg(y_0)=6(or)7)(or)((\deg(y_0)\le 5, \deg(x_i)\ge 6, \\ &\deg(x_j)\le 5), x_i \in C_{m1}, x_j \in C_{m2}))(Or)(j\ge 4, and \\ &(\deg(y_0)\le 5, \deg(x_i)\ge 3, \deg(x_j)\ge 3, \deg(x_s)\ge 3, x_i, \\ &x_j, x_s \in C_{m1})(or)(j=4\, and\ \deg(y_0)\ge 4, \deg(x_i)\ge 3, \\ &\deg(x_j)\ge 3, \deg(y_s)\ge 3, x_i, x_j \in C_{m1}, y_s \in C_{m2})] \end{aligned} \\ (s+1)-r & otherwise \end{cases}$$

ISSN: 0067-3064

Proof:

Case 1. Suppose H= $B_1\left(m_1, m_2\right)$

Then $\psi=\{C_{m1}, C_{m2}\}$ is a minimum 2-simple gc of H with y_0 as external vertex in H. Therefore $\eta_{2s}(H)=2$.

Case 2. When j =1 and let y_0 be the unique vertex is of degree >2 on $B_1\left(m_1, m_2\right)$ Then, we have three sub cases.

Subcase 2.1. When degree(y_0)=5

Consider $H^*=H-\{C_{m2}\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*)=(s+2)-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a=\psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+3)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+*2*) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+2)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq q-p-t_2+t=(p+1)-p-r+(s+2)=(s+3)$ $-r$. Thus $\eta_{2s}(H)=(s+3)-r$.

Subcase 2.2. When degree(y_0)=6 or 7

Consider $H^*=H-\{C_{m2}\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*)=(s+1)-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a=\psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H)=(s+2)-r$.

Subcase 2.3. When degree(y_0) ≥ 8

Consider $H^*=H-\{C_{m2}\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*)=s-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a=\psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H)=(s+1)-r$.

Case 3. When j =2 and let x_i be the only vertex is of degree >2 other than y_0 on $B_1\left(m_1, m_2\right)$. Suppose $x_i \in C_{m1}$, then there are two sub cases.

Subcase 3.1. When degree$(x_i) \leq 5$, then there are three sub cases.

Subcase 3.1.1. Suppose degree(y_0) =4 or 5

Similar to Subcase 2.1, the proof is the same..

Subcase 3.1.2. Suppose degree(y_0) =6 or 7

Similar to Subcase 2.2, the proof is the same.

Subcase 3.1.3. Suppose degree(y_0) ≥8

The proof is the same as in Subcase 2.3.

Subcase 3.2. When degree(x_i) ≥ 6, then there are two sub cases.

Subcase 3.2.1. Suppose degree(y_0) =4 or 5

The proof is the same as in Subcase 2.2.

Subcase 3.2.2. Suppose degree(y_0) ≥ 6

Similar to Subcase 2.3, the proof is the same.

Case 4. When j =3 and suppose y_0, x_i, x_j be the only vertices is of degree >2 on $B_1\left(m_1, m_2\right)$. Then, we have two cases.

Subcase 4.1. Suppose $x_i, x_j \in C_{m1}$, then there are three cases.

Subcase 4.1.1. When degree(x_i) ≤ 5, degree (x_j) ≤ 5 and degree(y_0) =4

Similar to Subcase 3.1.1. the proof is the same.

Subcase 4.1.2. When degree(x_i) ≥3, degree (x_j) ≥ 3, degree(y_0) = 5 (Or) either degree(x_i) ≥ 6 (or) degree (x_j) ≥ 6 and degree(y_0) =4 (or) 5

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+1)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 4.1.3. When degree(x_i) ≥3, degree (x_j) ≥ 3, degree(y_0) ≥ 6

Similar to Subcase 3.2.2, the proof is the same.

Subcase 4.2. Suppose $x_i \in C_{m1}$, $x_j \in C_{m2}$, then there are three cases.

Subcase 4.2.1. When degree(x_i) ≤ 5, degree (x_j) ≤ 5 and let s_1 be the number of pendant vertices and r_1 be the number of vertices is of degree ≥ 4 on sub tree rooted at x_j. Let the graph H_1 obtained from the cycle C_{m2} and its sub tree rooted at x_j. Since the unicyclic graph H_1 contains s_1 number of pendant vertices and r_1 vertices with the degree ≥ 4, By theorem 4.3, $\eta_{2s}(H_1) = (s_1+1) - r_1$. Then, we have three cases.

Subcase 4.2.1.1. When degree(y_0) =4 or 5

ISSN: 0067-3064

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1) - 1$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1) + 2 - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+2)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+2)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r$.

Subcase 4.2.1.2. When degree(y_0) =6 or 7

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1)$ vertices of degree $\geq$ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1) + 1 - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 4.2.1.3. When degree(y_0) ≥ 8

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1)$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1) - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.2.2. When degree(x_i) ≥ 6, degree (x_j) ≤ 5 and let the graph H_1 obtained from the cycle C_{m2} and its sub tree rooted at x_j. Since the unicyclic graph H_1 contains s_1 number of pendant vertices and r_1 be the number of vertices is of degree ≥ 4 on sub tree rooted at x_j. By theorem 4.3, $\eta_{2s}(H_1) = (s_1 + 1) - r_1$. Then, we have two cases.

Subcase 4.2.2.1. When degree(y_0) = 4 or 5

Similar to Subcase 4.2.1.2, the proof is the same.

ISSN: 0067-3064

Subcase 4.2.2.2. When degree(y_0) = 6 or 7

Similar to Subcase 4.2.1.3, the proof is the same.

Subcase 4.2.3. When degree(x_i) ≥ 6, degree (x_j) ≥ 6 and let P_1 and P_2 denote (x_i, y_0) and (y_0, x_j) section on $B_1(m_1, m_2)$. Then, we have two cases.

Subcase 4.2.3.1. When degree(y_0) = 4 or 5

Consider $H^* = H - \{P_1\} - \{P_2\}$ is a tree with pendant vertices s and $(r-1)$ vertices of degree ≥ 4. From theorem 4.2, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^* respectively. Then $\psi_a = \psi_0 \cup \{P_1\} \cup \{P_2\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \ge s$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.2.3.2. When degree(y_0) ≥ 6

Consider $H^* = H - \{P_1\} - \{P_2\}$ is a tree with pendant vertices s and $(r-1)$ vertices of degree ≥ 4. From theorem 4.2, $\eta_{2s}(H^*) = s - r - 1$. Let ψ_0 be the minimum 2-simple gc of H^* respectively. Then $\psi_a = \psi_0 \cup \{P_1\} \cup \{P_2\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \ge s$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Case 5. Suppose $j \ge 4$ and let $y_0, x_i, x_j, x_s(y_s)$ be the vertices is of degree >2 on $B_1(m_1, m_2)$. Then, we have three cases.

Subcase 5.1. When $x_i, x_j, x_s \in C_{m1}$ and degree(x_i) ≥3, degree (x_j) ≥ 3, degree (x_s) ≥ 3, then there are two cases.

Subcase 5.1.1. When degree(y_0) = 4 or 5

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+1)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 5.1.2. When degree(y_0) ≥ 6

ISSN: 0067-3064

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 5.2. When $j = 4, x_i, x_j \in C_{m1}, y_s \in C_{m2}$ and degree(y_0) ≥ 4,degree(x_i) ≥3, degree(x_j) ≥ 3, degree (y_s) ≥ 3. Let $P = \left(x_i, x_j\right)$ be a path in $B_1\left(m_1, m_2\right)$. Then, we have three sub cases.

Subcase 5.2.1. When (degree(x_i) = degree (x_j) = 3) (or) (degree(x_i) ≥ 5 and degree (x_j) = 3) or (degree(x_i) ≥ 5 and degree (x_j) ≥ 5)

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = \left(s+1\right) - r$. Let ψ_0 be the minimum 2-simple gc of H^* respectively. Then $\psi_a = \psi_0 \cup \{P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, $\left(s+1\right)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq \left(s+1\right)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 5.2.2. When degree(x_i) = degree (x_j) = 4

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = \left(s+3\right) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1, P_2 be a path in ψ_0 in which x_i, x_j are external. Then $\psi_a = \left(\psi_0 - P_1 - P_2\right) \cup \{P_1 \cup P \cup P_2\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, $\left(s+1\right)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq \left(s+1\right)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 5.2.3. When degree(x_i) ≥ 5 and degree (x_j) = 4

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = \left(s+2\right) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1 be a path in ψ_0 in which x_j external. Then $\psi_a = \left(\psi_0 - P_1\right) \cup \{P_1 \cup P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2) - r$.

Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H) = (s+2)-r$.

Subcase 5.3. When $j \geq 5$ and $x_i, x_j \in C_{m1}, y_s, y_l \in C_{m2}$, degree(y_0) $\geq$ 4, degree(x_i) $\geq$3, degree (x_j) $\geq$ 3, degree (y_s) $\geq$ 3, degree (y_l) $\geq$ 3. Let $P = (y_s, y_l)$ be a path in C_{m2}, then there are three cases.

Subcase 5.3.1. When degree (y_s) = degree (y_l) = 3 (Or) degree (y_s) $\geq$ 5 and degree (y_l) $\geq$ 5

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and r vertices of degree $\geq$ 4. From theorem 4.3, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^* respectively. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H) = (s+1)-r$.

Subcase 5.3.2. When degree (y_s) = 3 (or) degree (y_s) $\geq$ 5 and degree (y_l) = 4

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices $(s+1)$ and $(r-1)$ vertices of degree $\geq$ 4, so that $\eta_{2s}(H^*) = (s+2)-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1 be a path in ψ_0 in which is y_l external. Then $\psi_a = (\psi_0 - P_1) \cup \{P_1 \cup P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H) = (s+2)-r$.

Subcase 5.3.3. When degree (y_s) = degree (y_l) =4

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree$\geq$ 4, so that $\eta_{2s}(H^*) = (s+2)-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1, P_2 be a path in ψ_0 in which is y_s, y_l as external. Then $\psi_a = (\psi_0 - P_1 - P_2) \cup \{P_1 \cup P \cup P_2\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H) = (s+1)-r$.

Example 4.2. As an example, the bicyclic graph $B_1(m_1, m_2)$ shown in Figure 1.

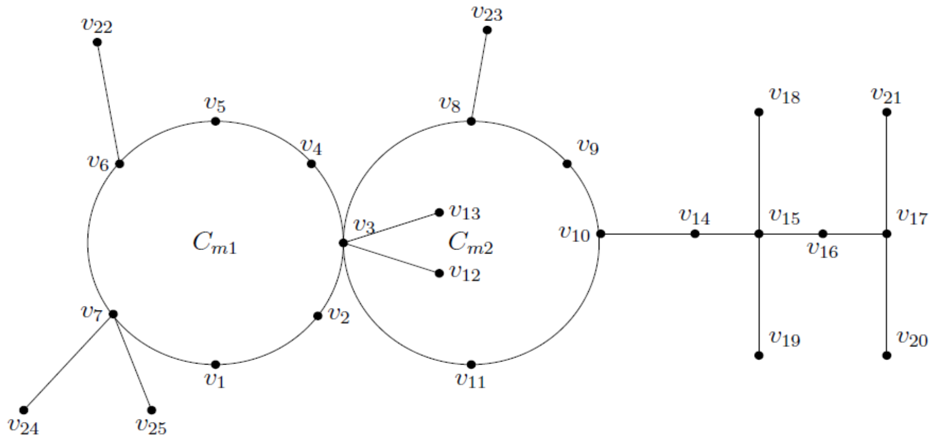

Figure 1. 2-simple gc on $B_1(m_1,\ m_2)$

Here p =25, q =26, s =10 and r =3. Then $\eta_{2s}(H)=\{(v_{22},v_6,v_5,v_4,v_3,v_{13}),(v_8,v_9,v_{10}),(v_{12},v_3,v_{11},v_{10},v_{14},v_{15},v_{16},v_{17},v_{20}),(v_3,v_8,v_{23}),(v_3,v_2,v_1,v_7,v_{25}),(v_{24}\ v_7,v_6),(v_{19},v_{15},v_{18}),(v_{17},v_{21})\}=8=(s+1)-r.$

Theorem 4.3. Let H be a bicyclic graph with s pendant vertices. Also let $B_2(m_1,m_2,r)$ be the unique bicycle in H and let j be the number of vertices of degree >2 on $B_2(m_1,m_2,r)$. Then

$$\eta_{2s}(H)=\begin{cases} 3 & if\ \ H=B_2(m_1,m_2,r) \\ (s+3)-r & \begin{array}{l} if\,\big[(j=2\ and\ \deg(x_1)\le 5,\ \deg(y_1)\le 5\ )(or) \\ (j=3\ and\ \deg(x_1)\le 5,\ \deg(y_1)\le 5,\deg(x_s)\le 5 \\ (or)(j=4\ and\ \deg(x_1)\le 5,\deg(y_1)\le 5 \\ \deg(x_s)\le 5,\deg(y_s)\le 5,x_s\in C_{m1},y_s\in C_{m2})\big] \end{array} \\ (s+2)-r & \begin{array}{l} if\,\big[(j=2\ and\ ((\deg(x_1)\ge 5,\deg(y_1)\le 6)(or)(\deg(x_1)\ge 6, \\ \deg(y_1)\ge 3))(j=3 and\,(\deg(x_1)\le 5,\deg(y_1)\ge 6,\deg(x_s)\le 5) \\ (either\ \deg(x_1)\ge 6\ (or)\ \deg(x_s)\ge 6\ and\ \deg(y_1)\le 5)) \\ (or)(\ j\ge 4,\deg(x_s)\ge 3,\deg(x_p)\ge 3,\deg(x_1)\ge 3, \\ \deg(y_1)\ge 5,x_s,x_p\in C_{m1})(or)(j=4,\deg(x_s)\le 5,\deg(x_1)\ge 5, \\ \deg(x_p)\le 5,\deg(x_1)\le 5,\deg(y_1)\ge 6,x_s\in C_{m_1},x_p\in C_{m_2})) \\ (or)(\ j\ge 5,\deg(x_s)\ge 3,\deg(x_p)\ge 3,\deg(x_l)\le 5, \\ \deg(x_1)\ge 3,\deg(y_1)\le 5,x_s,x_p\in C_{m1},x_l\in C_{m2})\big] \end{array} \\ (s+1)-r & otherwise \end{cases}$$

Proof: The cycles $C_{m1}=(x_0,x_1,..,x_{r-1},x_0)$, $C_{m2}=(y_0,y_1,..,y_0)$ and a path $P_r=(x_1\ ,x_m,\ldots,y_1)$ are the components of the Bicyclic graph $B_2(m_1,m_2,r)$.

Case 1. Suppose H= $B_2(m_1, m_2, r)$

Then $\psi_a = \{C_{m1}, C_{m2}, P_r\}$ is a minimum 2-simple gc of H. Therefore $\eta_{2s}(H) = 3.$

Case 2. Suppose j =2, then there are two cases.

Subcase 2.1. When $\text{degree}(x_1) \le 5$, then there are three subcases.

Subcase 2.1.1. When $\text{degree}(y_1) = 3$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices $(s+1)$ and r vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s+2) - r.$ Let ψ_0 be the minimum 2-simple gc of $H^*.$ Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+3) - r.$ Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+2)$ and $t_2 \le r.$ Hence $\eta_{2s}(H) \ge (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r.$

Subcase 2.1.2. When $\text{degree}(y_1) = 4(\text{or})5$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s+2) - r.$ Let ψ_0 be the minimum 2-simple gc of $H^*.$ Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le$ $(s+3) - r.$ Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+2)$ and $t_2 \le r.$ Hence $\eta_{2s}(H) \ge (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r.$

Subcase 2.1.3. When $\text{degree}(y_1) \ge 6$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s+1) - r.$ Let ψ_0 be the minimum 2-simple gc of $H^*.$ Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le$ $(s+2) - r.$ Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+1)$ and $t_2 \le r.$ Hence $\eta_{2s}(H) \ge (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r.$

Subcase 2.2. When $\text{degree}(x_1) \ge 6$, then there are three subcases.

Subcase 2.2.1. When $\text{degree}(y_1) = 3$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices $(s+1)$ and r vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s+1) - r.$ Let ψ_0 be the minimum 2-simple gc of $H^*.$ Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \le (s+2) - r.$ Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H) = (s+2)-r$.

Subcase 2.2.2. When $\text{degree}(y_1) = 4(\text{or})5$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s+1)-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H) = (s+2)-r$.

Subcase 2.2.3. When $\text{degree}(y_1) \geq 6$

Consider $H^* = H - \{C_{m2}\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = s-r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{C_{m2}\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H) = (s+1)-r$.

Case 3. Suppose j = 3 and let x_s, x_1, y_1 be the only vertices is of degree > 2 on $B_2(m_1, m_2, r)$. Suppose $x_s \in C_{m1}$, then we have two sub cases.

Subcase 3.1. When $\text{degree}(x_s) \leq 5$ and $\text{degree}(x_1) \leq 5$, then there three subcases.

Subcase 3.1.1. When $\text{degree}(y_1) = 3$

Similar to Subcase 2.1.1, the proof is the same.

Subcase 3.1.2. When $\text{degree}(y_1) = 4(\text{or})5$

Similar to Subcase 2.1.2, the proof is the same.

Subcase 3.1.3. When $\text{degree}(y_1) \geq 6$

Similar to Subcase 2.1.3, the proof is the same.

Subcase 3.2. When either $\text{degree}(x_s) \geq 6$ (or) $\text{degree}(x_1) \geq 6$ and suppose $\text{degree}(x_1) \geq 6$, then there are three subcases.

Subcase 3.2.1. When $\text{degree}(y_1) = 3$

Similar to Subcase 2.2.1, the proof is the same.

Subcase 3.2.2. When $\text{degree}(y_1) = 4(\text{or})5$

Similar to Subcase 2.2.2, the proof is the same.

Subcase 3.2.3. When $\text{degree}(y_1) \geq 6$

Similar to Subcase 2.2.3, the proof is the same.

ISSN: 0067-3064

Case 4. When $j \geq 4$ and let x_s, x_p be the only vertex is of degree > 2 other than x_1, y_1 on $B_2(m_1, m_2, r)$ is obtained by connecting two cycles. Then, we have four subcases.

Subcase 4.1. When $x_s, x_p \in C_{m1}$, degree$(x_s) \geq 3$, degree$(x_p) \geq 3$, degree$(x_1) \geq 3$, then there are three cases.

Subcase 4.1.1. When degree$(y_1) = 3$

Similar to Subcase 2.2.1, the proof is the same.

Subcase 4.1.2. When degree$(y_1) = 4$(or)5

Similar to Subcase 2.2.2, the proof is the same.

Subcase 4.1.3. When degree$(y_1) \geq 6$

Similar to Subcase 2.2.3, the proof is the same.

Subcase 4.2. When j =4 and $x_s \in C_{m1}$, $x_p \in C_{m2}$, then there are two cases.

Subcase 4.2.1. Suppose degree$(x_s) \leq 5$, degree$(x_p) \leq 5$, degree$(x_1) \leq 5$ and let the graph H_1 obtained from the cycle C_{m2} and its sub tree rooted at x_p. Since the unicyclic graph H_1 contains s_1 number of pendant vertices and r_1 be the number of vertices is of degree ≥ 4 on sub tree rooted at x_p. By theorem 4.3, $\eta_{2s}(H_1) = (s_1+1) - r_1$. Then, we have three subcases.

Subcase 4.2.1.1. When degree$(y_1) = 3$

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1) + 1$ and $(r - r_1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s - s_1) + 2 - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+2)$ pendant vertices and at most *r* vertices are internally two times. Therefore $t \geq (s+2)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r$.

Subcase 4.2.1.2. When degree$(y_1) = 4$(or)5

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1) - 1$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1) + 2 - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+2)$ pendant vertices and at most *r* vertices are

ISSN: 0067-3064

internally two times. Therefore $t \geq (s+2)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+3)-r$. Thus $\eta_{2s}(H)=(s+3)-r$.

Subcase 4.2.1.3. When degree$(y_1) \geq 6$

Consider $H^* = H-\{H_1\}$ is a unicyclic graph with pendant vertices $(s-s_1)$ and $(r-r_1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*)=(s-s_1)+1-(r-r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a=\psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H)=(s+2)-r$.

Subcase 4.2.2. When either degree$(x_s) \geq 6$ (or) degree$(x_1) \geq 6$ and degree(x_p) ≥ 6, suppose degree$(x_1) \geq 6$ and let the graph H_1 obtained from the cycle C_{m2} and its sub tree rooted at x_p. Since the unicyclic graph H_1 contains s_1 number of pendant vertices and r_1 be the number of vertices is of degree ≥ 4 on sub tree rooted at x_p. By theorem 4.3, $\eta_{2s}(H_1)=s_1-r_1$. Then, we have three subcases.

Subcase 4.2.2.1. When degree$(y_1)=3$

Consider $H^* = H-\{H_1\}$ is a unicyclic graph with pendant vertices $(s-s_1)+1$ and $(r-r_1)$ vertices of degree $\geq$ 4, so that $\eta_{2s}(H^*)=(s-s_1)+1-(r-r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a=\psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H)=(s+1)-r$.

Subcase 4.2.2.2. When degree$(y_1)=4$ (or) 5

Consider $H^* = H-\{H_1\}$ is a unicyclic graph with pendant vertices $(s-s_1)$ and $(r-r_1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*)=(s-s_1)+1-(r-r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a=\psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H) = (s+1)-r$.

Subcase 4.2.2.3. When degree$(y_1) \geq 6$

Consider $H^* = H - \{P\}$, where P denotes (y_1, x_l) section on $B_2(m_1, m_2, r)$. Clearly H^* is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4 so that, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1)-r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1)-r$. Thus $\eta_{2s}(H) = (s+1)-r$.

Subcase 4.3. When $j \geq 5$ and let x_s, x_p, x_l be the only vertices is of degree >2 other than x_1, y_1 on $B_2(m_1, m_2, r)$. Suppose, degree$(x_s) \geq 3$, degree$(x_p) \geq 3$, $x_s, x_p \in C_{m1}$ degree$(x_1) \geq 3$, $x_l \in C_{m2}$ and let the graph H_1 obtained from the cycle C_{m2} and its sub tree rooted at x_l. Since the unicyclic graph H_1 contains s_1 number of pendant vertices and r_1 be the number of vertices is of degree ≥ 4 on sub tree rooted at x_p. By theorem 4.3, $\eta_{2s}(H_1) = (s_1+1)-r_1$. Then, we have two cases.

Subcase 4.3.1. When degree$(x_l) \leq 5$, then there are three subcases.

Subcase 4.3.1.1. When degree$(y_1) = 3$

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)+1$ and $(r - r_1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s - s_1) +1-(r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2)-r$. Thus $\eta_{2s}(H) = (s+2)-r$.

Subcase 4.3.1.2. When degree$(y_1) = 4$(or)5

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1)-1$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1)+1 -(r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, $(s+1)$ pendant vertices and at most r

vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 4.3.1.3. When $\text{degree}(y_1) \geq 6$

Consider $H^* = H - \{H_1\}$ is a unicyclic graph with pendant vertices $(s - s_1)$ and $(r - r_1)$ vertices of degree ≥ 4. From theorem 4.3, $\eta_{2s}(H^*) = (s - s_1) - (r - r_1)$. Let ψ_0 and ψ_1 be the minimum 2-simple gc of H^* and H_1 respectively. Then $\psi_a = \psi_0 \cup \psi_1$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.3.2. When $\text{degree}(x_l) \geq 6$, then there are three subcases.

Subcase 4.3.2.1. When $\text{degree}(y_1) = 3$

Similar to Subcase 4.3.1.1. the proof is the same.

Subcase 4.3.2.2. When $\text{degree}(y_1) = 4 (\text{or}) 5$

Similar to Subcase 4.3.1.2. the proof is the same.

Subcase 4.3.2.3. When $\text{degree}(y_1) \geq 6$

Consider $H^* = H - \{P\}$, where P denotes (y_1, x_l) section on $B_2(m_1, m_2, r)$. Clearly H^* is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4 so that, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.4. Suppose $j \geq 6$ and let x_s, x_p, x_l, x_m be the only vertices is of degree >2 other than x_1, y_1 on $B_2(m_1, m_2, r)$. Suppose $x_s, x_p \in C_{m1}, x_l, x_m \in C_{m2}$ and let $P = (x_l, x_m)$ be a path in $B_2(m_1, m_2, r)$, then, we have three cases.

Subcase 4.4.1. Suppose $\text{degree}(x_l) = \text{degree}(x_m) = 3, (\text{Or})$ $\text{degree}(x_l) \geq 5,$ $\text{degree}(x_m) \geq 5$

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4 so that, $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$.

 ISSN: 0067-3064

Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.4.2. When degree$(x_l) = 3$ (Or) degree$(x_l) \geq 5$ and degree$(x_m) = 4$

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1 be a path in ψ_0 in which is x_m external. Then $\psi_a = (\psi_0 - P_1) \cup \{P_1 P\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 4.4.3. When degree(x_l)=degree$(x_m) = 4$

Consider $H^* = H - \{P\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, so that $\eta_{2s}(H^*) = (s+2) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_1, P_2 be a path in ψ_0 in which is x_l, x_m as external. Then $\psi_a = (\psi_0 - P_1 - P_2) \cup \{P_1 P P_2\}$ is a 2-simple gc of H, hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Example 4.4 As an example, the bicyclic graph $B_2(m_1, m_2, r)$ shown in Figure 2.

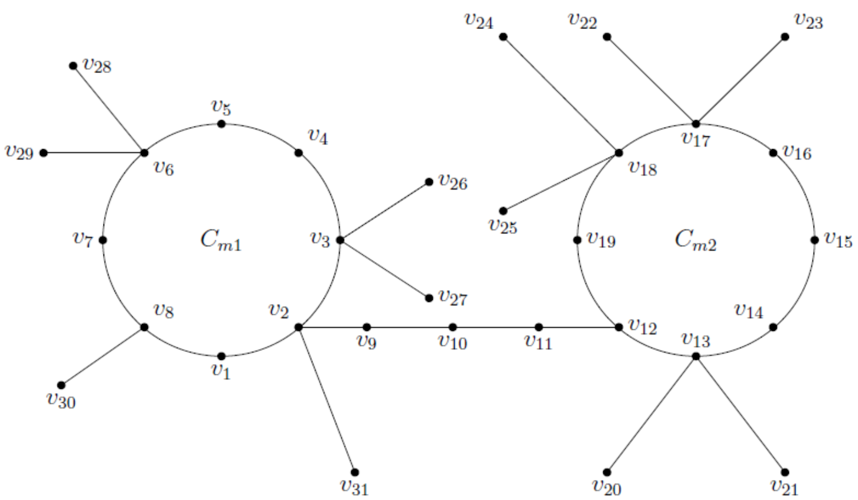

Figure 2. 2-simple gc on $B_2(m_1, m_2, r)$

Here $p = 31, q = 32, s = 12$ and $r = 6$, then $\eta_{2s}(H) = \{(v_{21}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{23}),$ $(v_{26}, v_3, v_{27}), (v_{28}, v_6, v_5, v_4, v_3, v_2, v_9, v_{10}, v_{11}, v_{12}.v_{13}, v_{20}), (v_{25}, v_{18}, v_{19}, v_{12}), (v_8, v_7, v_6$ $, v_{29})(v_{31}, v_2, v_1, v_8, v_{30}), (v_{22}, v_{17}, v_{18}, v_{24})\} = 7 = (s+1) - r.$

Theorem 4.5. Let H be a bicyclic graph with s pendant vertices. Also let $B_3(m_1, m_2, t)$ be the unique bicycle in H and let j be the number of vertices of degree >2 on $B_3(m_1, m_2, t)$, Then

$$\eta_{2s}(H) = \begin{cases} 3 & \text{if } H = B_3(m_1, m_2, t) \\ (s+3) - r & \text{if } [(j = 2 \text{ and } deg(u) \le 6, deg(v) \le 6)(\text{or})(j = 3, deg(x_i) \le 5(\text{or}) \\ & deg(w_i) \le 5,\ deg(u) = 3, deg(v) \le 6)(or)(j = 4,\ deg(x_k) \ge 3, \\ & deg(y_k) \ge 3, deg(x) = 4, deg(y) = 3)] \\ (s+1) - r & \text{if } [(j = 2 \text{ and } deg(u) \ge 7, deg(v) \ge 3)(\text{or})(j = 3 \text{ and } (deg(x_i) \ge 3 \\ & (\text{or})\, deg(w_i) \ge 3 \text{ and } deg(u) \ge 4, deg(v) \ge 4)\ (\text{or})\ (\text{either} \\ & deg(x_i) \ge 6\ (or)\ deg(w_i) \ge 6\, and\ deg(u) \ge 3, deg(v) \ge 3)(or) \\ & (\text{either } deg(x_i) \ge 3\ (or)\ deg(w_i) \ge 3, deg(u) \ge 7 \text{ and } deg(v) \ge 3)) \\ & (\text{or})(j = 4, \text{either } deg(x_i) \ge 3(or)\, deg(w_i) \ge 3, \text{either } deg(x_j) \ge 3 \\ & (\text{or})\, deg(w_j) \ge 3 \text{and } ((deg(u) \ge 3, deg(v) \ge 4)(\text{or})(deg(u) \ge 5, \\ & deg(v) \ge 5))(\text{or})(j \ge 5, \text{either } deg(x_i) \ge 3(\text{or})\, deg(w_i) \ge 3, \\ & \text{either } deg(x_j) \ge 3\ (\text{or})\ deg(w_j) \ge 3\ , \text{either } deg(x_k) \ge 3(\text{or}) \\ & deg(w_k) \ge 3, deg(v) \ge 4, deg(u) \ge 3, x_i, x_j, x_k \in P_1)] \\ (s+1) - r & \text{otherwise} \end{cases}$$

Proof: Let $P_1 = (u, x_1, \ldots, x_{m-1}, v), P_2 = (u, y_1, \ldots, y_{n-1}, v)$ and $P_3 = (u, w_1, \ldots, w_{t-1}, y)$ be the three internal disjoint paths of $B_3(m_1, m_2, t)$.

Case 1. When H= $B_3(m_1, m_2, t)$

Then $\psi = \{(u, x_1, .., x_{m-1}, v, y_{n-1}, y_n, .., y_1, u), (u, w_1, w_2, .., w_{t-1}), (w_{t-1}, v)\}$ is a minimum 2-simple gc of H so that $\eta_{2s}(H) = 3.$

Case 2. When j=2 and let P_{3t_1}, P_{3t_2} denote $(u, w_1 w_{t-1})$ and (w_{t-1}, v) section of P_3 respectively. Then, we have two cases.

Subcase 2.1. When degree$(u) \le 6$ and degree$(v) \le 6$, then there are three cases.

Subcase 2.1.1. When degree$(u) =$ degree$(v) = 3(\text{or})\ 5\ (\text{or})\ 6$

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥4, hence $\eta_{2s}(H^*) = (s+1) - r.$ Let ψ_0 be the minimum 2-simple

 ISSN: 0067-3064

gc of $H*$. Then $\psi_a = \psi_0 \cup \{P_{3t_1}, P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \le (s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+2)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r$.

Subcase 2.1.2. When $\text{degree}(u) = 4$ and $\text{degree}(v) = 3$ (or) 5 (or) 6

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+2) - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_u be a path in ψ_0 in which u is external. Then $\psi_a =$ $(\psi_0 - P_3) \cup \{P_u \cup P_{3t_1}, P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \le$ $(s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+2)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r$.

Subcase 2.1.3. When $\text{degree}(u) = 4$ and $\text{degree}(v) = 4$

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+3) - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_u, P_v be a paths in ψ_0 in which u and v are external. Then $\psi_a = (\psi_0 - P_u - P_v) \cup \{P_u \cup P_{3t_1}, P_v \cup P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \le (s+3) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+2)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+3) - r$. Thus $\eta_{2s}(H) = (s+3) - r$.

Subcase 2.2. When either $\text{degree}(u) \ge 7$ (or) $\text{degree}(v) \ge 7$ and suppose $\text{degree}(u) \ge 7$, then there are two cases.

Subcase 2.2.1. When $\text{degree}(v) = 3$ (or) $\text{degree}(v) \ge 5$

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Then $\psi_a = \psi_0 \cup \{P_{3t_1}, P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \le (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \ge (s+1)$ and $t_2 \le r$. Hence $\eta_{2s}(H) \ge (s+2) - r$. Thus $\eta_{2s}(H) =$ $(s+2) - r$.

Subcase 2.2.2. When $\text{degree}(v) = 4$

ISSN: 0067-3064

Consider $H^{*}=H-\{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^{*})=(s+1)-r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_v be a path in ψ_0 in which v is external. Then $\psi_a=(\psi_0-P_v)\cup\{P_{3t_1},P_{3t_2}\cup P_v\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H)\leq(s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t\geq(s+1)$ and $t_2\leq r$. Hence $\eta_{2s}(H)\geq(s+2)-r$. Thus $\eta_{2s}(H)=(s+2)-r$.

Case 3. When $j=3$ and either x_i (or) w_i be the only vertex is of degree >2 other than u and v. Let P_{2t_1} and P_{2t_2} denote $(u,y_1,....,y_{n-1})$ and (y_{n-1},v) section of P_2 respectively. Then, we have four cases.

Subcase 3.1. Suppose either degree$(x_i)\leq 5$ (or) degree$(w_i)\leq 5$, then we have two subcases.

Subcase 3.1.1. Suppose degree$(u)=3$ and degree$(v)=3$ (or) 5 (or) 6

Consider $H^{*}=H-\{P_2\}$ is a unicyclic graph with pendant vertices s and r vertices of degree $\geq$ 4, hence $\eta_{2s}(H^{*})=(s+1)-r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Then $\psi_a=\psi_0\cup\{P_{2t_1},P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H)\leq(s+3)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t\geq(s+2)$ and $t_2\leq r$. Hence $\eta_{2s}(H)\geq(s+3)-r$. Thus $\eta_{2s}(H)=(s+3)-r$.

Subcase 3.1.2. Suppose degree$(u)=3$ and degree$(v)=4$

Consider $H^{*}=H-\{P_2\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^{*})=(s+2)-r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_v be a path in ψ_0 in which v is external. Then $\psi_a=(\psi_0-P_v)\cup\{P_{2t_1},P_{2t_2}\cup P_v\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H)\leq(s+3)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+2) pendant vertices and at most r vertices are internally two times. Therefore $t\geq(s+2)$ and $t_2\leq r$. Hence $\eta_{2s}(H)\geq(s+3)-r$. Thus $\eta_{2s}(H)=(s+3)-r$.

Subcase 3.2. Suppose either degree$(x_i)\geq 3$ (or) degree$(w_i)\geq 3$, then we have three subcases.

Subcase 3.2.1. When degree$(u)=$ degree$(v)=4$

ISSN: 0067-3064

Consider $H^* = H - \{P_2\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+2) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_u, P_v be a paths in ψ_0 in which u and v are external. Then $\psi_a = (\psi_0 - P_u - P_v) \cup \{P_u \cup P_{2t_1}, P_v \cup P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 3.2.2. When $\text{degree}(u) = 4$ and $\text{degree}(v) \geq 5$

Consider $H^* = H - \{P_2\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_u be a path in ψ_0 in which u is external. Then $\psi_a = (\psi_0 - P_2) \cup \{P_u \cup P_{2t_1}, P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 3.2.3. When $\text{degree}(u) \geq 5$ and $\text{degree}(v) \geq 5$

Consider $H^* = H - \{P_2\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P_{2t_1}, P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 3.3. Suppose either $\text{degree}(x_i) \geq 6$ (or) $\text{degree}(w_i) \geq 6$, then we have three subcases.

Subcase 3.3.1. When $\text{degree}(u) = \text{degree}(v) = 3$ (or) $\text{degree}(u) \geq 5, \text{degree}(v) \geq 5$

Consider $H^* = H - \{P_2\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P_{2t_1}, P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 3.3.2. Suppose $\text{degree}(u)=3$ (or) $\text{degree}(u)\geq 5$ and $\text{degree}(v)=4$

Consider $H^{*}=H-\{P_2\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^{*})=(s+1)-r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_v be a path in ψ_0 in which v is external. Then $\psi_a=(\psi_0-P_v)\cup\{P_{2t_1},P_{2t_2}\cup P_v\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H)\leq(s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t\geq(s+1)$ and $t_2\leq r$. Hence $\eta_{2s}(H)\geq(s+2)-r$. Thus $\eta_{2s}(H)=(s+2)-r$.

Subcase 3.3.3. When $\text{degree}(u)=\text{degree}(v)=4$

Consider $H^{*}=H-\{P_2\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^{*})=(s+2)-r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_u,P_v be a paths in ψ_0 in which u and v are external. Then $\psi_a=(\psi_0-P_u-P_v)\cup\{P_u\cup P_{2t_1},P_v\cup P_{2t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H)\leq(s+2)-r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t\geq(s+1)$ and $t_2\leq r$. Hence $\eta_{2s}(H)\geq(s+2)-r$. Thus $\eta_{2s}(H)=(s+2)-r$.

Subcase 3.4. When $\text{degree}(u)\geq 7$ and either$\left(\text{degree}(x_i)\geq 3 \text{ (or) } \text{degree}(w_i)\geq 3\right)$, then there are two subcases.

Subcase 3.4.1. Suppose $\text{degree}(v)=3$ (or) $\text{degree}(v)\geq 5$

Similar to Subcase 3.3.1. the proof is the same.

Subcase 3.4.2. Suppose degree(v)=4

The proof is the same as in Subcase 3.3.2.

Case 4. When $j=4$ and either x_i,x_j (or) w_i,w_j be the only vertices is of degree > 2 other than u and v. Let P_{3t_1} and P_{3t_2} denote $(u,w_1,..,w_{t-1})$ and (w_{t-1},v) section of P_3 respectively. Suppose $x_i,x_j\in P_1$ (or) $x_i,\in P_1$ and $x_j\in P_2$, then there are four cases.

Subcase 4.1. When $\text{degree}(x_i)\leq 5$, $\text{degree}(x_j)\leq 5$ and $\text{degree}(u)=\text{degree}(v)=3$

Similar to Subcase 2.2.1, the proof is the same.

Subcase 4.2. When $\text{degree}(x_i)\geq 3$, $\text{degree}(x_j)\geq 3$, $\text{degree}(u)=3$ (or) $\text{degree}(u)\geq 5$ and $\text{degree}(v)=4$

Similar to Subcase 2.2.2, the proof is the same.

Subcase 4.3. When $\text{degree}(u)=\text{degree}(v)=4$, $\text{degree}(x_i)\geq 3$ and $\text{degree}(x_j)\geq 3$

 ISSN: 0067-3064

Consider $H^{*} = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-2)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^{*}) = (s+2) - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Let P_u, P_v be a paths in ψ_0 in which u and v are external. Then $\psi_a = (\psi_0 - P_u - P_v) \cup \{P_u \cup P_{3t_1}, P_v \cup P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Subcase 4.4. When $\text{degree}(x_i) \geq 3, \text{degree}(x_j) \geq 3$, $\text{degree}(v) \geq 5$ and $\text{degree}(u) \geq 5$ (Or) either $\text{degree}(u) \geq 7\,(or)\ \text{degree}(v) \geq 7, \text{degree}(x_i) \geq 3$ and $\text{degree}(x_j) \geq 3$

Consider $H^{*} = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, hence $\eta_{2s}(H^{*}) = s - r$. Let ψ_0 be the minimum 2-simple gc of H^*. Then $\psi_a = \psi_0 \cup \{P_{3t_1}, P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+2) - r$. Furthermore, for any 2-simple gc ψ_a of H, (s+1) pendant vertices and at most r vertices are internally two times. Therefore $t \geq (s+1)$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+2) - r$. Thus $\eta_{2s}(H) = (s+2) - r$.

Case 5. When $j \geq 5$ and either x_i, x_j, x_s (or) w_i, w_j, w_s be the only vertices is of degree >2 other than u and v. Let P_{3t_1} and P_{3t_2} denote $(u, w_1, ..., w_{t-1})$ and (w_{t-1}, v) section of P_3 respectively. Then, we have two cases.

Subcase 5.1. When $j \geq 5$ and Suppose $x_i, x_j, x_s \in P_1$ then there are three cases.

Subcase 5.1.1. When $\text{degree}(x_i) \geq 3$, $\text{degree}(x_j) \geq 3$, $\text{degree}(x_s) \geq 3$, $\text{degree}(u) = 3$ (or) $\text{degree}(u) \geq 5$ and $\text{degree}(v) = 3$ (or) $\text{degree}(v) \geq 5$

Similar to Subcase 2.2.1, the proof is the same.

Subcase 5.1.2. When $\text{degree}(x_i) \geq 3$, $\text{degree}(x_j) \geq 3$, $\text{degree}(x_k) \geq 3$, $\text{degree}(v) = 4$ and $\text{degree}(u) = 3\,(\text{or})\ \text{degree}(u) \geq 5$

Similar to Subcase 2.2.2, the proof is the same.

Subcase 5.1.3. When $\text{degree}(x_i) \geq 3$, $\text{degree}(x_j) \geq 3$, $\text{degree}(x_k) \geq 3, \text{degree}(u) = 4$ and $\text{degree}(v) = 4$

Similar to Subcase 4.3, the proof is the same.

Subcase 5.2. When $j \geq 5$ and let the paths P_{3t_1} and P_{3t_2} denote $(u, w_1, .., w_{t-1})$ and (w_{t-1}, v) section of P_3. Suppose $x_i, x_k \in P_1, y_j \in P_2$ (or) $x_i, x_k \in P_1$ and $y_j, y_l \in P_2$, Then, we have three cases.

Subcase 5.2.1. Suppose either (degree(x_i) ≥ 3, degree(x_j) ≥ 3, degree(y_j)) ≥ 3) (or) (degree(x_i) ≥ 3, degree (x_j) ≥ 3, degree (y_j) ≥ 3, degree (y_l) ≥ 3) and degree(u) $= 3$, degree(v) $= 3$ (or) degree(u) ≥ 5, degree(v) ≥ 5

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and r vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = s - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Then $\psi_a = \psi_0 \cup \{P_3\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 5.2.2. Suppose either (degree(x_i) ≥ 3, degree(x_j) ≥ 3, degree(y_j)) ≥ 3) (or) (degree(x_i) ≥ 3, degree (x_j) ≥ 3, degree (y_j) ≥ 3, degree (y_l) ≥ 3) and degree(u) $= 3$ (or) degree(u) ≥ 5, degree(v) $= 4$

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_v be a path in ψ_0 in which v external. Then $\psi_a = (\psi_0 - P_v) \cup \{P_3 \cup P_v\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Subcase 5.2.3. Suppose either (degree(x_i) ≥ 3, degree(x_j) ≥ 3, degree(y_j)) ≥ 3) (or) (degree(x_i) ≥ 3, degree (x_j) ≥ 3, degree (y_j) ≥ 3, degree (y_l) ≥ 3) and degree(u) $= 4$, degree(v) $= 4$

Consider $H^* = H - \{P_3\}$ is a unicyclic graph with pendant vertices s and $(r-1)$ vertices of degree ≥ 4, hence $\eta_{2s}(H^*) = (s+1) - r$. Let ψ_0 be the minimum 2-simple gc of $H*$. Let P_u, P_v be a paths in ψ_0 in which u and v are external. Then $\psi_a = (\psi_0 - P_u - P_v) \cup \{P_u \cup P_{3t_1}, P_v \cup P_{3t_2}\}$ is a minimum 2-simple gc of H and hence $\eta_{2s}(H) \leq (s+1) - r$. Furthermore, for any 2-simple gc ψ_a of H, s pendant vertices and at most r vertices are internally two times. Therefore $t \geq s$ and $t_2 \leq r$. Hence $\eta_{2s}(H) \geq (s+1) - r$. Thus $\eta_{2s}(H) = (s+1) - r$.

Example 4.6. As an example, the bicyclic graph $B_3(m_1, m_2, t)$ shown in Figure 3.

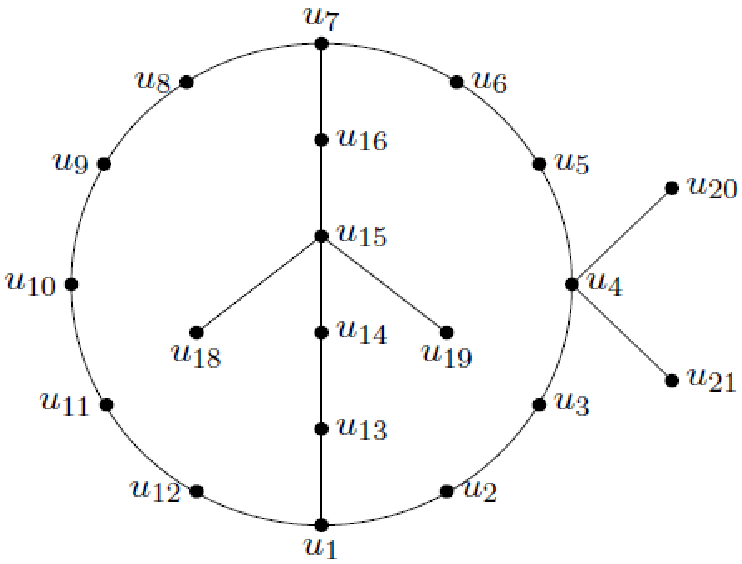

Figure 3. 2-Simple graphoidal cover on $B_3(m_1, m_2, t)$

Here $p = 21, q = 22, s = 4$ and $r = 2$, then $\eta_{2s}(H) = \{(u_4, u_5, v_6, u_7, v_8, u_9, u_{10}, u_{11}, u_{12}, u_1, u_2, u_3, u_4), (u_{20}, u_4, u_{21}), (u_1, u_{13}, u_{14}, u_{15}, u_{19}), (u_{18}, u_{15}, u_{16}, u_7) = 4 = (s+2) - r.$

5. Discussions

The remarkable progress in science and technology, particularly within the realm of medical sciences, has significantly enhanced our ability to save lives. Nevertheless, it is frustrating to observe the shortcomings in effectively tracing contacts of infected individuals to manage communicable diseases. The recent pandemic has underscored that comprehensive contact tracing (or) examining the full spectrum of individuals connected to an infected person is crucial for breaking the transmission chain and facilitating early treatment for both the infected and those at risk.

In a graph theory model, each individual is represented as a vertex and their interactions are depicted as edges. For instance, in a communicable disease network (in Figure 4), let's consider a scenario where person v_{25} (x) is initially infected. The disease spreads from x to individuals v_{21}, v_4, v_{11}, v_7 and others. To effectively contain the outbreak and provide treatment to those infected, we can break down the larger network into smaller groups. Starting with the initial infected person, x, we can segment the remaining individuals into manageable groups. By closely monitoring these smaller units, we can better control the spread of the disease.

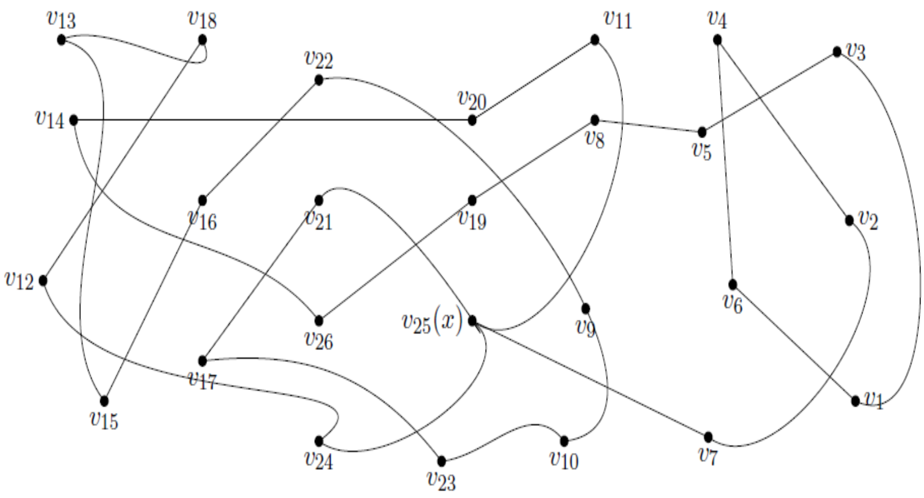

Figure 4. Communication disease network.

Using graph decomposition, we can visualize a cyclic structure that contains two or more small groups sharing one or more individuals, as illustrated in the accompanying Figure 5. This decomposition model facilitates the creation of a web that traces the immediate contacts of the infected person (x), helping to identify potential secondary infections and the extent of disease transmission originating from a single infected individual.

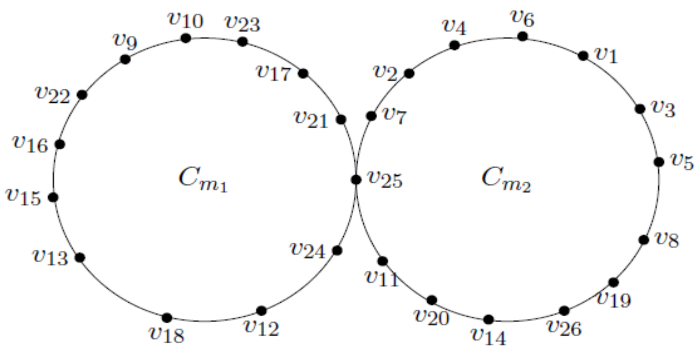

Figure 5. Bicyclic structure of communication disease network

6. Conclusion

Transportation networks, encompassing roads, railways, and airways, are intricate systems connecting nodes and edges, enabling the smooth flow of people, goods, and services, thereby fostering economic progress. However, these networks grapple with significant challenges, including traffic congestion, environmental harm, substantial maintenance expenses, and susceptibility to disruptions that can cripple the system.

A 2-simple graphoidal cover simplifies transportation networks by breaking them into manageable components, making rerouting in emergencies more efficient and reducing overall complexity. However, its application in large, dense networks can be computationally demanding and may conflict with practical requirements for redundancy. By minimizing the impact of disruptions on the entire system, smaller paths help maintain network stability.

Bicyclic graphs play a vital role in enhancing transportation networks by providing redundancy and fault tolerance. With multiple routes available, they ensure continuous flow even during disruptions caused by accidents, construction, or natural disasters. These graphs help reduce bottlenecks and delays while offering a modular structure that supports cost-effective and scalable design and expansion. As a result, bicyclic graphs are an invaluable tool for creating resilient and efficient transportation systems.

Bicyclic graphs combined with a 2-simple graphoidal cover enhance the efficiency and resilience of transportation networks. Bicyclic graphs provide redundancy, enabling smooth traffic rerouting during disruptions, while the 2-simple graphoidal cover ensures all edges are covered, leaving no route unaccounted for. This approach facilitates easy expansion, as the modular structure of bicyclic graphs supports the addition of new routes without major reconfiguration. Furthermore, it minimizes reliance on centralized control, leading to cost-effective and distributed management. The scalability of this method allows small, localized networks to grow seamlessly into larger systems by incorporating multiple bicyclic subgraphs.

7. References

[1] B Acharya and E Sampathkumar. Graphoidal covers and graphoidal covering number of a graph. *Indian J. pure appl. Math*, 18(10):882–890, 1987.

[2] BD Acharya. Further results on the graphoidal covering number of a graph. *Graph Theory News Lett.*, 17(4):1–3, 1988.

[3] C Pakkiam and S Arumugam. On graphoidal graphs of a tree. *Indian J. Pure Appl. Math.*, 21(12):1055–1058, 1990.

[4] C Pakkim and S Arumugam. The graphoidal covering number of unicyclic graphs. *Indian J. Pure Appl. Math*, 23(2):141–143, 1992.

[5] E Sampathkumar. Semigraphs and their applications. Technical report, 2000.

[6] Frank Harary. Covering and packing in graphs, i. *Annals of the New York Academy of Sciences*, 175(1):198–205, 1970.

[7] Frank Harary. *Graph theory*. Addison-Wesley, Reading, MA, 1969.

[8] I Sahul Hamid and Arumugam, S Simple graphoidal covers in a graph. *J. Comb. Math. Comb. Comput.*, 64:79–95, 2008.

[9] Indra Rajasingh and P Roushini Leely Pushpam. Equality of graphoidal and acyclic graphoidal covering number of a graph. *Tamkang Journal of Mathematics*, 37(2):175– 183, 2006.

[10] K Nagarajan, A Nagarajan, and S Somasundaram. 2-Graphoidal Path Covers. *Int. J. Appl. Math.*, 21(4):615–628, 2008.

[11] K Nagarajan, A Nagarajan, and S Somasundram. m-graphoidal path covers of a graph. In *Proceedings of the Fifth International Conference on Number Theory and Smarandache Notions*, pages 58–67, 2009.

[12] K.Ratan Singh and P K Das. On graphoidal covers of bicyclic graphs. In *Int. Math. Forum*, volume 5, pages 2093–2101, 2010.

[13] P. K. Das and K. Ratan Singh. On 2-graphoidal covering number of a graph. *Int. J. Pure Appl. Math.*, 72(2):125–135, 2011.

[14] S Arumugam, B Devadas Acharya, and E Sampathkumar. Graphoidal covers of a graph: a creative review. In *Proc. Natl. Work. Graph Theory its Appl. Manonmaniam Sundaranar Univ. Tirunelveli, Eds. S. Arumugam, BD Acharya E. Sampathkumar, Tata McGraw Hill*, pages 1–28, 1996.

[15] S Arumugam, Indra Rajasingh, and P Roushini Leely Pushpam. Graphs whose acyclic graphoidal covering number is one less than its maximum degree. *Discrete Mathematics*, 240(1-3):231–237, 2001.

[16] S. Arumugam and C. Pakkiam. On the graphoidal covering number of a graph. *Indian J. Pure Appl. Math.*, 20(4):330–333, 1989.

[17] S. Arumugam and I. Sahul Hamid. Simple acyclic graphoidal covers in a graph. *Australas. J. Comb.*, 37:243–255, 2007.

[18] S. Arumugam and J. Suresh Suseela. Acyclic graphoidal covers and path partitions in a graph. *Discrete Math.*, 190(1-3):67–77, 1998.

[19] Venkat Narayanan, Suresh Suseela, and Rukhmoni Kala. On 2 acyclic simple graphoidal covering of bicyclic graphs. *J. Math. Comput. Sci.*, 10(3):606–632, 2020.

[20] Venkat Narayanan, Suresh Suseela, and Rukhmoni Kala. On 2-simple graphoidal cover of a graph. *AIP Conference Proceedings*, 2277(4):100010–1–100010–10, 2020.

[21] Venkat Narayanan, Suresh Suseela, and Rukhmoni Kala. Simple graphoidal covering number of product of graphs. *International Journal of Pure and Applied Mathematics*, 111(2):233–248, 2016.

[22] Venkat Narayanan, Suresh Suseela, and Rukhmoni Kala. Simple graphoidal covers on digraphs. *International Journal of Pure and Applied Mathematics*, 118(2):239–253, 2018.